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A photonic equation of motion is proposed which is the scalar product of four-vectors and therefore a Lorentz invariant. A photonic equation of motion, which has not been heretofore established in quantum electrodynamics (QED), would capture the quantum nature of light but yet not have the standard field-operator form, thereby making practical calculations easier to perform. The equation of motion proposed here is applied to the Lamb shift. No divergences exist, and the result agrees with the observed Lamb shift for the $1S_{1/2}$ state of hydrogen within experimental error.

Remarkably the photon does not have an established equation of motion (EOM), although finding a correct photonic EOM has been an active field of research in quantum optics for many years [1]. This problem is important for conceptual reasons – if the photon is really a particle, why does it not have an EOM? – and practical reasons – the field-operator equations of standard quantum electrodynamics (QED) often will not give up practical computational results in absence of serious approximations. Here we propose an EOM which is the scalar product of four-vectors and therefore a Lorentz invariant. We test our proposed EOM by calculating the Lamb shift.

We require that the electromagnetic continuity equation be written as the scalar product of four-vectors in analogy to the material continuity equation, which is well known to be the scalar product of the four-gradient and the four-current [2]. This enables us to infer a four-gradient which is special for electromagnetic phenomena. Then the photonic EOM is assumed to be the scalar product of the electromagnetic four-gradient and posited four-spinors which are written to insure the transverse nature of the propagation of a free photon. It turns out that that the four-spinors are the same ones using Pauli's vector whose scalar product with the electron four-momentum gives up Dirac's equation. Pace reader! We have shown previously [3] that the spin-statistics theorem [4] is not violated. The spin statistics theorem states that microscopic causality is violated if the commutation rules are scrambled – anti-commutators used for bosons or commutators used for fermions. We show in [3] that mass-0 particles are an exception to the theorem. The physical reason seems transparent: photons define the light cone such that photonic motion outside the light cone, which violates microscopic causality, cannot occur notwithstanding our choice of quantization rules. Physically the use of Dirac spinors preserves the transversality of photonic propagation – only two components of the spin-1 particle are observed in nature – and are its Lorentz invariant expression – in standard QED the Coulomb gauge is used to suppress one of the three components of the spin-1 particle, a procedure which is not manifestly Lorentz invariant. For these reasons, even though Dirac spinors are used, we may quantize the radiation field using commutators rather than anti-commutators without violating the spin-statistics theorem, and photons obey Bose-Einstein statistics as observed in nature.

The electromagnetic continuity equation for free fields can be written as the scalar product of four-vectors and therefore as a Lorentz invariant,

$$(\frac{1}{c}\frac{\partial}{\partial t},\vec{\nabla}-\frac{e}{mc^2}\vec{E},\vec{H})\cdot(cu,\frac{c}{4\pi}\vec{E}\times\vec{H})=0\ , \qquad (1)$$

where $u=\frac{1}{8\pi}(E^2+H^2)$. Notice that Eq. (1) differs from the standard result by the replacement, $\vec{\nabla}\rightarrow\vec{\nabla}-\frac{e}{mc^2}\vec{E},\vec{H}$, where the expression $\vec{E},\vec{H}$ means either the electric field or the magnetic field. This replacement is justified mathematically because the scalar product of either the electric field or the magnetic field and the electromagnetic current, $\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$, vanishes such that in either case we recover the standard electromagnetic continuity equation for the free fields,

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0. \tag{2}$$

Electric, magnetic EOM's are derived for the photon from the scalar products respectively of the electric, magnetic four-gradients and four-spinors which we will define shortly. This procedure is based on the success of writing the EOM for a relativistic electron – Dirac's equation – as the scalar product of the electron's four-momentum and four-spinors. Since this presentation of Dirac's equation does not to our knowledge appear anywhere in the literature, we give it here. Then it will be clear how to proceed with the photonic EOM's.

Dirac's pair of first-order equations are the scalar products of the four-momenta, $(\frac{E_L - V}{c}, i\hbar\vec{\nabla})$, $(\frac{E_S - V}{c}, i\hbar\vec{\nabla})$, and of vectors which we herewith posit as four-spinors, $(\psi_L, \vec{\sigma}\psi_S)$, $(\psi_S, \vec{\sigma}\psi_L)$, such that we obtain,

$$[(E_L - V)/c]\psi_L + i\hbar\vec{\sigma}\cdot\vec{\nabla}\psi_S = 0 \tag{3a}$$

$$[(E_S - V)/c]\psi_S + i\hbar\vec{\sigma}\cdot\vec{\nabla}\psi_L = 0 \quad . \tag{3b}$$

where $\vec{\sigma}$ is Pauli's vector. Recall that the scalar product of four-vectors is always a Lorentz invariant, here the zeroes on the right sides of Eqs. (3), which are easily recognized as Dirac's equations on using the definitions $E_L = E - mc^2$ and $E_S = E + mc^2$. This derivation uses the proof given in Morse and Feshbach [2] that a posited four-vector, here the four-spinor, is indeed a four vector if its scalar product with a known four-vector, here the four momentum, gives a Lorentz invariant, here the zero on the right side of Eq. (3a) or Eq. (3b), which is known from the identification of the left sides with Dirac's equations.

Notice that our presentation of Dirac's equation as a pair of scalar products of four-vectors and therefore as a pair of Lorentz invariants proves the covariance of his equation in a single step. This eliminates the need for a second step to prove the covariance of his wave equation [5] when the standard presentation [5] is used, which is to present the operator as the scalar product of four-vectors $(\gamma_0, \vec{\gamma})$ and $(\frac{E-V}{c}, i\hbar\vec{\nabla})$ to give $\frac{\gamma_0}{c}(E-V)+i\hbar\vec{\gamma}\cdot\vec{\nabla}$, which, operating on his four component wave function, gives the Lorentz invariant mc times his four-component wave function, where $\gamma_0 = \beta$ and $\vec{\gamma} = \beta\vec{\alpha}$ using the Dirac $\vec{\alpha}$ and β matrices.

Similarly we find the scalar products of the electric and magnetic four-gradients in succession with the photonic four-spinors $(\psi_{E,H}, \vec{\sigma}\varphi_{E,H})$, $(\varphi_{E,H}, \vec{\sigma}\psi_{E,H})$. First-order electric and magnetic EOM's follow. We present the stationary result in the form of electric and magnetic Helmholtz equations which are obtained from the first-order equations by elimination of one component in favor of the other. (The same equation is obtained for either elimination in the case of photons, just as in the mass-0 Dirac equation.) The Helmholtz equations are,

$$\{\nabla^2 + \frac{\omega^2}{c^2} - \frac{e}{mc^2}[\vec{\nabla}\cdot\vec{E} + 2\vec{E}\cdot\vec{\nabla} + i\sigma\cdot(\vec{\nabla}\times\vec{E}) - \frac{e}{mc^2}E^2]\}\psi_E = 0 \qquad \text{(4a)}$$

$$\{\nabla^2 + \frac{\omega^2}{c^2} - \frac{e}{mc^2}[\vec{\nabla}\cdot\vec{H} + 2\vec{H}\cdot\vec{\nabla} + i\sigma\cdot(\vec{\nabla}\times\vec{H}) - \frac{e}{mc^2}H^2]\}\psi_H = 0\,. \qquad \text{(4b)}$$

where we have used the identity,

$(\vec{\sigma}\cdot\vec{\nabla})(\vec{\sigma}\cdot\vec{E},\vec{H}) = \vec{\nabla}\cdot\vec{E},\vec{H} + i\vec{\sigma}\cdot(\vec{\nabla}\times\vec{E},\vec{H})$. .

Using $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}e\vec{j}$, where $\vec{j}$ is the current, the fifth term on the left side of of Eq. (4b) is responsible for the Lamb shift. We calculate the Dirac current $\vec{j} = c\psi^{+}\vec{\alpha}\psi$, where ψ is Dirac's four-component wave function formed from ψ_L and ψ_S given by Eq. (3), for the $1S_{1/2}$ and $2S_{1/2}$ states of a hydrogen-like ion, obtaining for the former state,

$$\vec{j} = \hat{i}j_x + \hat{j}j_y = \frac{c}{2\pi}(-\hat{i}y + \hat{j}x)\frac{G_{-1}(r)F_{-1}(r)}{r^3} \tag{5a}$$

$$\frac{4\pi e^2}{mc^3}i\vec{\sigma}\cdot\vec{j} = -\frac{4e^2(Z\alpha)(Z/a_0)^3}{mc^2}e^{-2(Z/a_0)r}M(\theta,\phi) \tag{5b}$$

$$M(\theta,\phi) = \mathrm{i}\begin{pmatrix} 0 & \mathrm{j_x} - \mathrm{i}\,\mathrm{j_y} \\ \mathrm{j_x} + \mathrm{i}\,\mathrm{j_y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sin\theta e^{-i\phi} \\ -\sin\theta e^{i\phi} & 0 \end{pmatrix} \tag{5c}$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant, $a_0 = \frac{\hbar^2}{me^2}$ is the Bohr radius, the reduced radial wave functions for the large and small components are $G_{-1}(r)$ and $F_{-1}(r)$ respectively, and we have kept only the leading terms in powers of α. Notice that the current is automatically transverse such that the Coulomb gauge condition $\vec{\nabla}\cdot\vec{A}=0$ is satisfied because $\vec{\nabla}\cdot\vec{j}=0$.

The eigenvalues of $M(\theta,\phi)$ are purely imaginary. Choice of the negative imaginary eigenvalue gives Eq. (4b) an absorptive "potential" such that the photonic wave ψ_H is absorbed as it passes through the atom. We conclude that the Lamb shift is given by the absorbed energy, which is,

$$E=\frac{1}{8\pi(2\pi^2)}\int_0^\infty dk k^2(\hbar\omega)\psi_k^2 P_a(k)\ , \qquad (6)$$

where $k = \omega/c$ and ψ_k is the radial wave function in momentum space, which is obtained from the Fourier transform of the $1S_{1/2}$ or $2S_{1/2}$ radial wave function.

The absorption probability $P_a(k)$ is calculated using standard theory for scattering from an absorptive potential [6],

$$P_a(k)=\frac{1}{4}\sum_{\ell\ell' m}(1-e^{-4\mu_{\ell\ell' m}})\ , \qquad (7)$$

where $\mu_{\ell\ell' m}$ is the imaginary part of the phase shift and for Z=1 we have assumed that the Born theory is valid. Expanding the exponential in Eq. (7) we obtain,

$$P_a(k) = \frac{2}{3}\sum_{\ell\ell' m}\mu^r_{\ell\ell'}[\delta_{\ell\ell'} - (2\ell'+1)(-1)^m \begin{pmatrix} \ell & 2 & \ell' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \ell & 2 & \ell' \\ m & 0 & -m \end{pmatrix}] \quad (8a)$$

$$\mu^r_{\ell\ell'} = k\int_0^\infty drr^2 j_{\ell'}(kr)U(r)j_\ell(kr) \quad (8b)$$

$$U(r) = \frac{4e^2(Z\alpha)(Z/a_0)^3}{mc^2} e^{-2(Z/a_0)r} \ . \quad (8c)$$

We repeat this calculation for the $2s_{1/2}$ state. Figs. 1-2 show E versus ℓ for the two states. The $2s_{1/2}$ calculation converges very slowly and we have cut the series off at $\ell_{\max} = 23$ due to numerical inaccuracies in our program to calculate higher-order spherical Bessel functions. The converged energy for the $1s_{1/2}$ state and the incompletely converged energy for the $2s_{1/2}$ state are given in Table 1 with comparisons to experimental and standard QED results. Our result for the $1s_{1/2}$ state agrees with the observed result within experimental error. Our Lamb shifts are smaller than those of standard QED calculations. This is possibly due to our use of target states unperturbed by the presence of the photon in the scattering calculation.

Our successful calculation of the Lamb shift suggests that our postulated EOM's for the photon [Eqs. (4)] are correct. The conceptual picture which emerges from the present theory however seems quite different from that of either established QED [7] or of controversial self-field electrodynamical theories [8]. The present theory converges with photonic frequency [Eq. (6)] because of the natural cut off imposed by the momentum distribution of the atomic state. The dominant interaction in the present theory occurs in the EOM for the photon [Eq. (4b)- Eqs. (5)] rather than in the EOM for the electron as in standard QED. This interaction has a form similar to that postulated by Pauli for the electron-magnetic field interaction. The electron's permanent magnetic moment can interact with anapplied magnetic field, as treated by Pauli and later by Dirac. According to the present theory the Dirac current, which arises from the electron's permanent magnetic moment, can interact with the atom's quantum mechanical electromagnetic field and cause it to be absorbed, thereby increasing the energy of the atom.

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Table 1. Calculated and experimental Lamb shifts.

	Present theory	QED [9}	Observed
$1S_{1/2}$	0.2688 cm^{-1}	0.2803 cm^{-1}	0.262 ± 0.038 [10]
$2S_{1/2}$	975.4 MHz	1074 MHz	1057.77 ± 0.10 [11]

Figure Captions

Fig. 1. Energy absorbed [Eq. (6)] by hydrogen in the $1S_{1/2}$ state versus photon partial wave ℓ.

Fig. 2. Energy absorbed [Eq. (6)] by hydrogen in the $2S_{1/2}$ state versus photon partial wave ℓ.

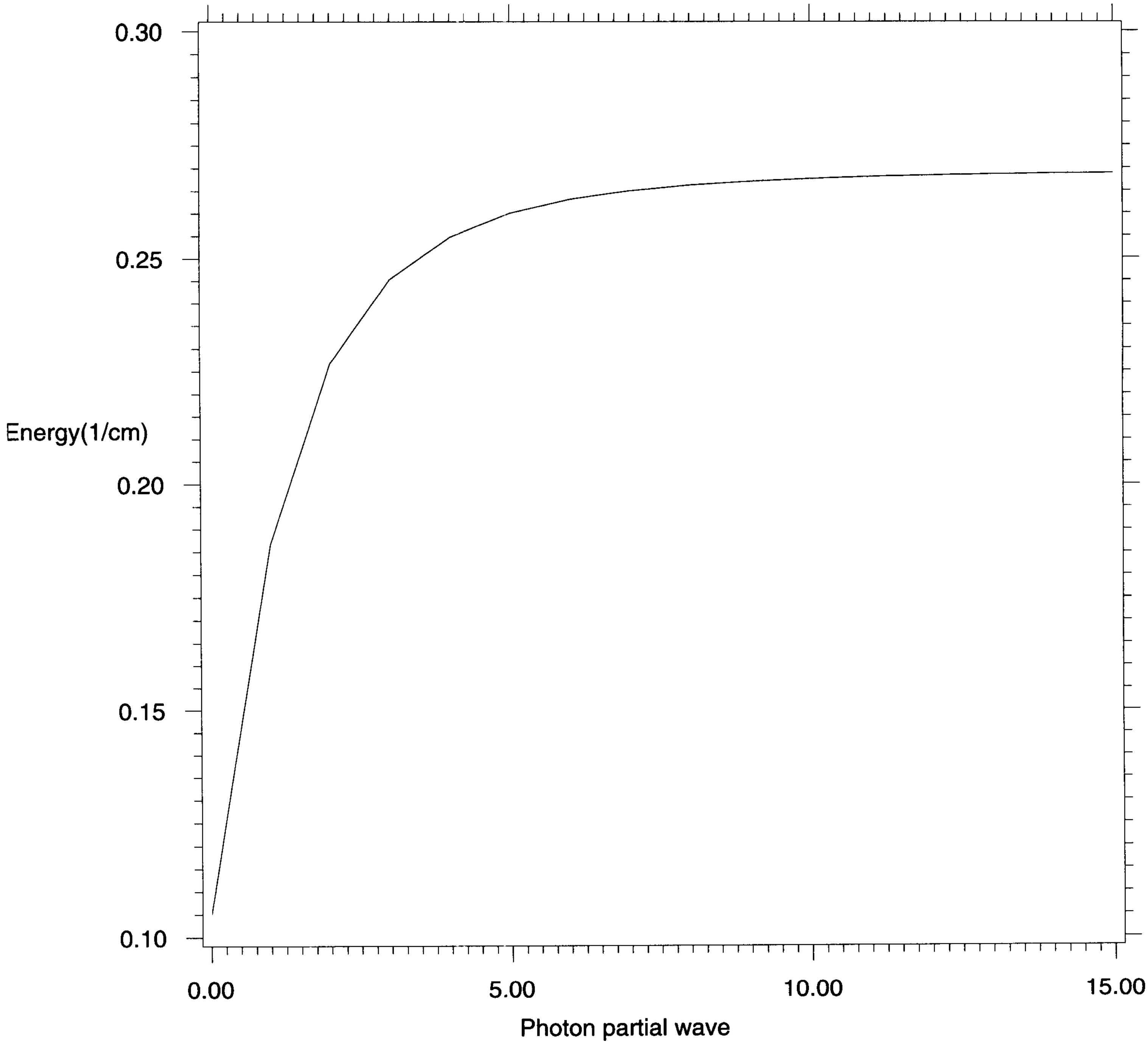
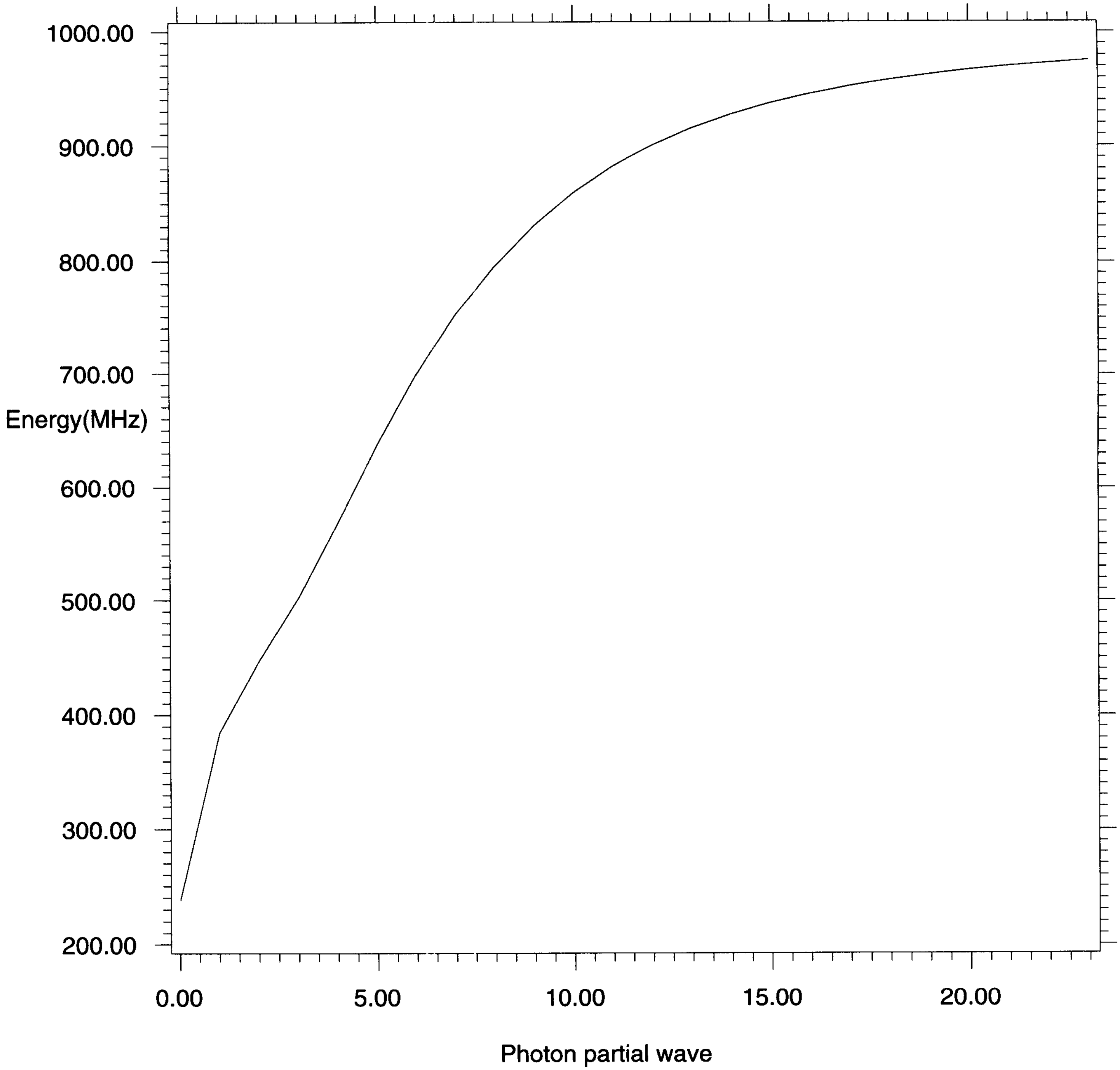

0.30
0.25
0.20
0.15
0.10
Energy(1/cm)
0.00
5.00
10.00
15.00
Photon partial wave

Fig. 1



1000.00
900.00
800.00
700.00
Energy(MHz)
600.00
500.00
400.00
300.00
200.00
0.00
5.00
10.00
15.00
20.00
Photon partial wave

Fig. 2